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VIBRATIONS OF ROLLING STOCK AND A THEOREM OF KRONECKER†

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The problem of the vibrations of a one-dimensional chain of $n+1$ linearly connected material points, under the action of a constant force F applied suddenly to the end point, is considered and the exact solution is analysed. It is proved that the supremum of the connecting force between the end point and its predecessor is equal to $(2Fn)/(n+1)$ if and only if the number of points is either prime or a power of 2. The behaviour of the supremum if this condition is not satisfied is demonstrated in the special case of six points. A discussion of the relationship with vibrations of the corresponding continuous medium and with the convergence of the method of straight lines is presented.

A SIMPLE and widely used model of a train is a collection of $n+1$ ($n \geq 1$) material points placed along a straight line and connected in series (see for example [1]). For simplicity, we shall assume that all the points have the same mass M and that the connections are linearly elastic with the same stiffness coefficient c. Denote the displacement of the jth point ($j=0, 1, \ldots, n$) from its initial, unloaded equilibrium position by $y_j = y_j(t)$, and the elastic force between the jth and $(j-1)$th points by $\sigma_j = c(y_j - y_{j-1})$, $j = 1, \ldots, n$.

Let us suppose that, beginning at a time $t=0$, a force $f(t)$ is applied to the 0th point (the "locomotive"). The functions $\sigma_j(t)$ will then be solutions of the system of equations

$$M\sigma_j^{\cdot\cdot}(t) = c(\sigma_{j+1} - 2\sigma_j + \sigma_{j-1}), \quad j = 1, \ldots, n$$
$$\sigma_0 := -f(t); \quad \sigma_{n+1} := 0 \tag{1}$$

with zero initial conditions.

If n is large, one often replaces this discrete model by a continuous one, i.e. system (1) is replaced by the wave equation

$$M\sigma_{tt}(x, t) = cH^2\sigma_{xx}(x, t) \tag{2}$$

where H is the initial distance between the material points, which is assumed to be independent of j, and $\sigma(x, t) = \sigma_j(t)$ if $x = jH$. Equation (2) is to be solved with the following initial data/boundary conditions:

$$\sigma(0, t) = -f(t), \quad \sigma(nH, t) = 0 \tag{3}$$

Such problems are solved by the method of characteristics.

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One is naturally interested in the relationship between the solutions of problems (1) and (2), (3). The problem may be formulated more clearly by fixing some $l>0$, introducing a new variable $\xi = l/(nH)x \in [0, l]$ and setting $nM/l = \rho$, $cl/n = E$, $l/n = h$ (this is the distance between the points on the ξ axis). Then system (1) becomes

$$\rho\sigma_j^{\cdot\cdot} = E\frac{\sigma_{j+1} - 2\sigma_j + \sigma_{j-1}}{h^2} \quad (j = 1, \ldots, n)$$

$$\sigma_0 = -f(t); \quad \sigma_{n+1} = 0 \tag{4}$$

On the other hand, application of the same transformation to Eq. (3) gives

$$\rho\sigma_{tt} = E\sigma_{\xi\xi}, \quad 0 \leqslant x \leqslant l, \quad 0 \leqslant t < \infty \tag{5}$$

Equation (5) is obtained from (4) by letting $n\to\infty$, $h\to 0$ while keeping $nh = l$, ρ, E constant.

Thus, our problem is to determine the relationship between discrete and continuous models of linear elastic longitudinal vibrations of a continuous medium. However, observing that system (4) is obtained from (5) by introducing the standard finite-difference approximation of the derivative $\sigma_{\xi\xi}$, one essentially arrives at the question: is it possible to use the method of straight lines to solve Eq. (5) approximately?

The method of straight lines is known to be applicable in problems of this type with suitable assumptions [2, 3]; indeed, some of the pioneers of mechanics (Lagrange, Rayleigh and Joukowski) concerned themselves with the natural transition from the discrete to the continuous model and vice versa. For these reasons, the passage from model (1) to model (2) is a widely used strategem in computational mathematics. However, the essential differences between these models must also be kept in mind.

Let us consider the simplest case: $f(t) = -F$ ($F = \text{const}>0$) (the train is set in motion by a constant pulling force). Then the solution of problem (2), (3) is a step function, which may be written

$$\sigma(x, t) = F\theta\left(nH \arcsin\left|\sin\left(\frac{\pi}{2n}\sqrt{\frac{c}{M}}\,t\right)\right| - x\right) \tag{6}$$

where $\theta(t)$ is the Heaviside unit function. Clearly, $0 \leqslant \sigma(x, t) \leqslant F$ for all $t \in R_+$, $x \in [0, nH]$.

On the basis of these and some other approximate arguments, some authors (e.g. [1, 4]) have assumed that when $f(t) = -F$ an analogous inequality holds for all components of the solution of system (1), provided only that n is large enough. This is in fact confirmed by a simple analysis of the solution of system (1) for $n = 1$:

$$\sigma_1(t) = {}^1/_2 F\,(1 - \cos\sqrt{2c/M}\,t)$$

However, direct computations for large $n = 40$, 80, 120 have shown that the values of $\sigma_j(t)$ at certain times t_j^* (different for different js) may considerably exceed F (by a few dozen percent)—a circumstance of immediate practical concern when one is dealing, say, with actual rolling stock.

Our aim in this note is to study this problem.

When $f(t) = -F$ system (1) can be solved exactly (see [4, p. 281]):

$$\sigma_j(t) = \frac{F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi k}{n+1}\,j\,\operatorname{ctg}\frac{\pi k}{2(n+1)}\right)(1 - \cos\omega_k t) \tag{7}$$

$$\omega_k = 2\sqrt{\frac{c}{M}}\sin\frac{\pi k}{2(n+1)}, \quad k = 1, \ldots, n$$

Formula (7) immediately yields upper and lower limits that are valid for all $t>0$:

$$-\frac{2F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi k}{n+1}j\right)_{(-)}\operatorname{ctg}\frac{\pi k}{2(n+1)}\leqslant\sigma_j(t)\leqslant$$
$$\leqslant\frac{2F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi k}{n+1}j\right)_{(+)}\operatorname{ctg}\frac{\pi k}{2(n+1)} \tag{8}$$
$$a_{(+)}=a\theta(a)=\begin{cases}a, & a\geqslant 0,\\ 0, & a<0,\end{cases}\quad a_{(-)}=a[\theta(a)-1]=\begin{cases}0, & a\geqslant 0\\ |a|, & a<0\end{cases}$$

Recall that numbers $\beta_1, \dots, \beta_n$ are said to be linearly independent over the field of rational numbers Q if, whenever $r_1\beta_1+\dots+r_n\beta_n=0$, where $r_k\in Q$, all the r_ks must vanish.

Theorem 1. If the numbers

$$\beta_k=\sin\frac{\pi k}{2(n+1)},\quad k=1,\dots,n \tag{9}$$

are linearly independent over Q, then for any $T\geqslant 0$ the limits (8) are the best possible in the interval $T\leqslant t<\infty$, i.e., for every $j=1,\dots,n$,

$$\sup_{t\geqslant T}\sigma_j(t)=\frac{2F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi k}{n+1}j\right)_{(+)}\operatorname{ctg}\frac{\pi k}{2(n+1)} \tag{10}$$

$$\inf_{t\geqslant T}\sigma_j(t)=-\frac{2F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi k}{n+1}j\right)_{(-)}\operatorname{ctg}\frac{\pi k}{2(n+1)} \tag{11}$$

and moreover if $n>1$ the supremum (10) is not achieved and if $j>1$ the same is true of the infimum (11) [when $j=1$ the infimum (11) is achieved only at $t=0$].

The proof of this theorem (and of Theorem 2, see below) are postponed to the Appendix. We will content ourselves here with the observation that the proof of Theorem 1 relies on a classical theorem of Kronecker on diophantine inequalities.

One consequence of this result is that the solution (6) (as $t\to+\infty$) is by no means a uniform asymptotic limit of the solution (7) as $n\to\infty$ if the frequencies ω_k are linearly independent over Q—at any rate, unless the data are somehow averaged over the x axis.

The criterion for linear independence of the numbers (9) yields the following theorem.

Theorem 2. The numbers (9) are linearly independent if and only if $n+1$ is either a prime number or a power 2^N, where $N=2, 3, \dots$.

Corollary. If $n+1\geqslant 3$ and $n+1$ is prime or $n+1=2^N$ ($N\geqslant 2$), then Eqs (10) and (11) are true.

In some cases inequalities (8) may be simplified. Thus, if $j=1$ it is always true that $\sin(\pi k/n+1)j>0$, so that by using the standard formula

$$\sum_{r=0}^{m-1}\cos(\alpha+rb)=\cos\left(\alpha+\frac{m-1}{2}b\right)\sin m\frac{b}{2}\Big/\sin\frac{b}{2} \tag{12}$$

one can show that the upper limit in (8) is

$$\frac{2F}{n+1}\sum_{k=1}^{n}\sin\frac{\pi k}{n+1}\operatorname{ctg}\frac{\pi k}{2(n+1)}=\frac{2F}{n+1}\sum_{k=1}^{n}\left(1+\cos\frac{\pi k}{n+1}\right)=\frac{2n}{n+1}F \tag{13}$$

while the lower limit vanishes. Thus, if $j=1$ we have simply

$$0\leqslant\sigma_1(t)\leqslant\frac{2n}{n+1}F \tag{14}$$

Similarly, if $j=n$ the same formula (12) enables us to show that

$$\frac{2F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi n}{n+1}k\right)_{(+)}\operatorname{ctg}\frac{\pi k}{2(n+1)}=$$

$$=\frac{2F}{n+1}\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\sin\left(\frac{2r+1}{n+1}\pi\right)\operatorname{ctg}\frac{2r+1}{2(n+1)}\pi=F$$

$$\frac{2F}{n+1}\sum_{k=1}^{n}\left(\sin\frac{\pi n}{n+1}k\right)_{(-)}\operatorname{ctg}\frac{\pi k}{2(n+1)}=\frac{2F}{n+1}\sum_{r=0}^{\left[\frac{n}{2}\right]-1}\times$$

$$\times\sin\left(\frac{2r+1}{n+1}\pi\right)\operatorname{ctg}\frac{2r+2}{2(n+1)}\pi=\frac{n-1}{n+1}F$$

so that if $j=n$ inequalities (8) become

$$-\frac{n-1}{n+1}F\leqslant\sigma_n(t)\leqslant F \tag{15}$$

Estimates (14) and (15) are the best possible if $n+1$ is prime or $n+1=2^N$ $(N\geqslant 2)$.

Obviously, although $\sigma_1(t)$ is always less than $2F$, it may approach as close to $2F$ as desired for large n of the above type. On the other hand, it is always true that $|\sigma_n(t)|\leqslant F$.

We will now need the formula

$$\sum_{k=1}^{n}\sin\frac{\pi k}{n+1}j\operatorname{ctg}\frac{\pi k}{2(n+1)}=n-j+1\ (n=1,2,\ldots;\ j=1,\ldots,n)$$

When $j=1$ this is equivalent to formula (13); for $j>1$ it may be proved by induction on j, using Eq. (12). It follows from this equation that the sum of the right-hand sides of (10) and (11) is $2(n-j+1)F(n+1)$, i.e. only one of these numbers needs to be evaluated. For relatively small n, direct evaluation of trigonometric functions yields the following limits, which are the best possible:

$$n=1:\ 0\leqslant\sigma_1(t)\leqslant F$$

$$n=2:\ 0\leqslant\sigma_1(t)\leqslant{}^4/_3F;\ \ -{}^1/_3F<\sigma_2(t)<F$$

$$n=3:\ 0\leqslant\sigma_1(t)<{}^3/_2F;\ \ -0{,}2071F<\sigma_2(t)<1{,}2071F;\ \ -0{,}5F<$$
$$<\sigma_3(t)<F$$

$$n=4:\ 0\leqslant\sigma_1(t)<1{,}6F;\ \ -0{,}2944F<\sigma_2(t)<1{,}4944F$$
$$-0{,}4944F<\sigma_3(t)<1{,}2944F;\ \ -0{,}6F<\sigma_4(t)<F$$

Theorem 1 does not apply when $n=5$ $(n+1=6)$, because, by Theorem 2, the numbers (9) will then be linearly dependent over Q. Indeed, it can be verified that when $n+1=6$

$$\beta_1 + \beta_3 - \beta_5 = 0 \tag{16}$$

but the numbers β_1, β_2, β_3, β_4 are linearly independent.

Let us analyse the case $n+1 = 6, j = 1$. By (13) and (14), the right-hand side of (7) is equal to

$$\frac{F}{6}\left[5 - \left(1 + \frac{\sqrt{3}}{2}\right)\cos\omega_1 t + \frac{3}{2}\cos\omega_2 t - \cos\omega_3 t - \frac{1}{2}\cos\omega_4 t - \right.$$
$$\left. - \left(1 - \frac{\sqrt{3}}{2}\right)\cos(\omega_1 t + \omega_3 t)\right] \tag{17}$$

A direct check shows that if $0 \leqslant a \leqslant 1$

$$\max_{u,\, v \in R} (-\cos u - \cos v - a\cos(u + v)) = 2 - a$$

and this maximum is reached when $\cos u = \cos v = -1$. It thus follows from Kronecker's theorem that the supremum (which is not reached) of (17) is

$$\frac{F}{6}\left[5 + 2 - \left(1 - \frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} + \frac{3}{2} + \frac{1}{2}\right] = \frac{8 + \sqrt{3}}{6} F = 1{,}622F$$

Thus, when $n+1 = 6$ exact evaluation improves the general estimate (14) by only 2.7%.

Other values of j, n may be considered similarly, but as n increases the analytical difficulties of an exact treatment increase rapidly, and it becomes more reasonable to use (7) and actually calculate the required quantities. It would be interesting to have a rigorous proof of some general properties of the functions σ_j, e.g. when the mass of the 0th point (the "locomotive") exceeds the mass M of each of the other points.

Of course, these considerations do not mean that the method of straight lines is not legitimate, since it is assumed that the time interval is fixed as n increases. When the method is applied to our problem, however, the periodic solution is approximated by almost periodic functions, and then convergence over a finite interval need not imply uniform convergence for all $t > 0$. On the contrary, on passing from system (1) to Eq. (2) one loses certain fine effects ("bursts"), though the transition is perfectly justifiable if one is only interested in average quantities.

In the language of mechanics, what we have just said means that when analysing the so-called "local properties" of a one-dimensional continuous medium, one cannot treat the medium as the limiting case of a linear chain of point masses, obtained when the number of points increases without limit.

It is natural to ask how long the time interval must actually be for "bursts" to be detectable. Our results imply that the presence of such effects depends only on the arithmetic properties of the number of points (prime or a power of two) but not on the masses M or stiffnesses c. However, a glance at the formula for the frequencies ω_k will reveal that these bursts appear at times that depend on $\sqrt{(c/M)}$. The practical range within which M and c may vary is quite large (e.g., M may vary from atomic masses to the mass of a railway carriage). We have calculated that at M and c values corresponding to railway rolling stock such bursts of forces, exceeding F by approximately 30% appear even over a 15 second time interval.

Finally, we note that the results remain valid if the pulling force $f(t)$ increases to F not in a stepwise fashion but continuously, and fairly rapidly; this is the case, for example, if $f(t) = F(1 - e^{\varkappa} t)$ with a sufficiently large exponent $\varkappa > 0$.

We wish to thank Yu. V. Kuz'min for discussing the results.

APPENDIX

Proof of Theorem 1. That the supremum (10) and infimum (11) are not reached follows from the linear independence of the frequencies ω_k over Q; for $n>1$ and any $j=1, \ldots, n$, at least two of the numbers $\sin(\pi k/n+1)j$ $(k=1, \ldots, n)$ do not vanish, while if $j>0$ they are not all of the same sign. The fact that the limits (8) are the best possible when $\sigma_j(t)$ is defined for all $t \in R$ by Eq. (7) follows from the following version of Kronecker's theorem (see, for example, [5]): if real numbers $\omega_1, \ldots, \omega_n$ are linearly independent over Q, then for any $\delta>0$ and any real numbers $\lambda_1, \ldots, \lambda_n$ there exist integers m_k such that the system of inequalities

$$|\omega_k t^* - \lambda_k - 2\pi m_k| < \delta$$

is solvable for t^*. Noting that the functions $\sigma_j(t)$ are linearly independent, we arrive at (10) and (11).

Proof of Theorem 2. Set $\alpha_k = \cos[\pi k/2(n+1)]$ $(k=1, \ldots, n)$. . Then $\beta_k = a_{n+1-k}$ and it will suffice to check the numbers $\alpha_1, \ldots, \alpha_n$ for linear independence over Q

We have $\alpha_k + i\beta_k = \zeta^k$, where $\zeta = \exp(2\pi i/m)$, $m = 4(n+1)$.

Before proceeding with the proof we will recall some facts from algebra. Let P, U be number fields (subfields of the field C of complex numbers), there $P \subseteq U$, i.e. U is an extension of P. Numbers $u_1, \ldots, u_n \in U$ are said to be linearly independent over P if any equality $p_1 u_1 + \ldots + p_n u_n = 0$ (where all p_j are elements of P) implies that all the p_js vanish. This enables one to define the dimensionality of U over P, also known as the degree of the extension and denoted by $[U:P]$. The degree has the following property: if P, U, V are number fields such that $P \subseteq U \subseteq V$, then

$$[V:P] = [V:U] \cdot [U:P] \tag{18}$$

We can now proceed to the proof proper. By the theory of the cyclotomic equation (see, e.g. [6]), the set V of all numbers of the form $r_0 + r_1\zeta + \ldots + r_{m-1}\zeta^{m-1}$ (where all the r_js are in Q) form a field, and moreover $[V:Q] = \varphi(m)$, where $\varphi(m)$ is Euler's function, that is, the number of natural numbers not greater than m and prime to m. [Thus, $\varphi(1)=1$, $\varphi(3)=2$, $\varphi(4)=2$, $\varphi(5)=4$, $\varphi(6)=2$ and so on. We will also need the following property of Euler's function: if $m_1 m_2 \in N$ are coprime, then $\varphi(m_1, m_2) = \varphi(m_1)\varphi(m_2)$.] It also follows from the general theory that ζ is a root of some polynomial of degree $\varphi(m)$ with coefficients in Q, defined uniquely by these conditions apart from a constant factor; but it is not a root of any polynomial of degree less than $\varphi(m)$ over Q.

Let U denote the set of all real numbers in V. Clearly, U is a field and $\alpha_k = 1/2(\zeta^k + \zeta^{m-k}) \in U$ for all k. Since

$$\zeta^k = \left(\frac{\zeta^k + \zeta^{m-k}}{2}\right) + \left(\frac{\zeta^k + \zeta^{m-k}}{\zeta - \zeta^{m-1}}\right)\frac{\zeta - \zeta^{m-1}}{2}$$

and the fractions in parentheses are real, it follows that every number in V can be expressed as $u_1 + u_2(\zeta - \zeta^{m-1})$, where $u_1, u_2 \in U$. Thus $[V:U] \leqslant 2$. But $U \neq V$, so $[V:U]=2$ and it follows from (18) that $[U:Q] = 1/2\varphi(m)$. Therefore, if $n > 1/2\varphi(m)$, the numbers $\alpha_1, \ldots, \alpha_n$ are linearly dependent over Q.

We will now prove that if $n+1$ is neither a power of two nor a prime number, then this inequality must be true. Indeed, suppose that $n+1$ is neither a power of two nor a prime. Then $n+1 = 2^N q$, where $N = 0, 1, \ldots$ and q is odd. But then

$$\begin{aligned} {}^1/_2\varphi(m) &= {}^1/_2\varphi(2^{N+2}q) = {}^1/_2\varphi(2^{N+2})\,\varphi(q) = 2^N\varphi(q) \leqslant 2^N(q-1) = \\ &= n+1-2^N \leqslant n \end{aligned}$$

If q is not a prime, the first inequality in this chain of inequalities is strict; but if q is a prime, then $N \geqslant 1$ and therefore the second inequality is strict, as required.

We can now show that if $n+1$ is either a power of two or a prime, the numbers $\alpha_1, \ldots, \alpha_n$ are linearly independent. Suppose we are given a non-trivial linear relation

$$r_1\alpha_1 + \ldots + r_n\alpha_n = 0 \tag{19}$$

where $r_1, \ldots, r_n \in Q$, but not all the r_js are zero. Since $\alpha_k = 1/2(\zeta^k + \zeta^{-k})$, multiplication of (19) by $2\zeta^n$ will show that ζ is a root of the polynomial

$$g(x) = r_1(x^{n+1} + x^{n-1}) + r_2(x^{n+2} + x^{n-2}) + \ldots + r_n(x^{2n} + 1)$$

Let $n+1 = p$ be an odd prime. Then

$$\varphi(m) = \varphi(4p) = 2(p-1) = 2n$$

On the other hand, it is obvious from the equalities

$$\zeta^{4p} - 1 = (\zeta^2 + 1)(\zeta^{2p} - 1)(\zeta^{2(p-1)} - \zeta^{2(p-2)} + \ldots - \zeta^2 + 1) = 0 \tag{20}$$

that $s(\zeta) = 0$, where $s(x) = x^{2(p-1)} - x^{2(p-2)} + \ldots - x^2 + 1$. Since the degree of $s(x)$ is $\varphi(m)$, the polynomial $g(x)$, which is of degree $\leqslant \varphi(m)$, must be proportional to $s(x)$. But that is impossible, since the coefficient of x^n in $s(x)$ is 1 or -1, while that in $g(x)$ is 0.

Finally, if $n+1 = 2^N$, $N \geqslant 1$, we have

$$\varphi(m) = \varphi(2^{N+2}) = 2^{N+1} = 2(n+1)$$

Thus no polynomial over Q of degree less than $2(n+1)$, in particular $g(x)$, can have ζ for a root.

This completes the proof of Theorem 2.

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